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CONSTRUCTION OF FOLD MAP OF LENS SPACE $L(p, 1)$ WHERE SINGULAR SET IS A TORUS

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1. Introduction

Throughout the report, all manifolds and maps are differentiable of class C^{∞}. Let $f : M \to \mathbb{R}^p$ be a map of a closed n-dimensional manifold M into $\mathbb{R}^p$ $(n \geq p)$. We denote by $S(f)$ the set of points in M where the rank of the differential of f is strictly less than p. We say that $S(f) \subset M$ is a *singular set* of f and $f(S(f)) \subset \mathbb{R}^p$ is a *contour* of f.

Let $f : M \to \mathbb{R}^3$ be a map of a closed connected oriented 3-dimensional manifold M into $\mathbb{R}^3$. For any $q \in S(f)$ of $f : M \to \mathbb{R}^3$, if we can choose local coordinates (u_1, u_2, u_3) centered at q and (v_1, v_2, v_3) centered at $f(q)$ respectively such that f has the following form:

$$(v_1 \circ f, v_2 \circ f, v_3 \circ f) = (u_1, u_2, u_3^2), \tag{1.1}$$

then we call f a *fold map*. It is known that if $f : M \to \mathbb{R}^3$ is a fold map, then $S(f)$ is a closed orientable surface (not necessary connected) and $f|S(f) : S(f) \to \mathbb{R}^3$ is an immersion. If $f|S(f)$ is an immersion with normal crossings, we call f a *stable fold map*.

Eliashberg [2] showed that if a closed surface V splits M into two manifolds M_1, M_2 with $\partial M_1 = \partial M_2 = V$, then there exists a fold map $f : M \to \mathbb{R}^3$ such that $S(f) = V$. Here, M_1 and M_2 are not necessary connected. In this report, we apply Eliashberg's theorem to a lens space $L(p, 1)$ and construct a stable fold map $f : M \to \mathbb{R}^3$ such that $S(f) = T^2$ is a Heegaard surface of $L(p, 1)$ $(p \geq 2)$.

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2. Description of a stable fold map

In this section, we explain a method to depict a stable fold map $f : M \to \mathbb{R}^3$. In the following, we assume that M is a closed connected oriented 3-dimensional manifold and that $\mathbb{R}^3$ and $\mathbb{R}^2$ are oriented.

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For a stable fold map $f : M \to \mathbb{R}^3$ such that $S(f) = V$ and $M = M_1 \cup_V M_2$, we remark that $f|M_1$ and $f|M_2$ are immersions and extensions of $f|V$. We assume that $f|M_1$ is an orientation preserving immersion and $f|M_2$ is an orientation reversing immersion. The orientation on M_1 induces the orientation on V as follows. For $q \in V$, let $\{n_1, n_2, nt_3\}$ be the basis of $T_q(M_1)$ which defines the orientation on M_1 and n_1 the outward normal vector. Then the orientation on $V = \partial(M_1)$ is defined by $\{n_2, n_3\}$.

By Bruce and Kirk's theorem [1], there exists an orthogonal projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ such that $\pi \circ f|V : V \to \mathbb{R}^2$ is a stable map. It is well-known that a stable map satisfies the following properties.

Proposition 2.1 ([3]). *A smooth map $g : N \to \mathbb{R}^2$ of a closed surface N into $\mathbb{R}^2$ is a stable map if and only if the following conditions are satisfied.*

(1) *For every $q \in S(g)$, there exist local coordinates (u_1, u_2) and (v_1, v_2) around q and $g(q)$ respectively such that one of the following holds:*
   (i) $(v_1 \circ g, v_2 \circ g) = (u_1, u_2^2)$, *$q$: fold point,*
   (ii) $(v_1 \circ g, v_2 \circ g) = (u_1, u_2^3 - u_1u_2)$, *$q$: cusp point.*

(2) *If q is a cusp point of g, then $g^{-1}(g(q)) \cap S(g) = \{q\}$,*

(3) *$g|S(g) \setminus \{$set of cusp points of $g\}$ is an immersion with normal crossings.*

In the following, we set $f_V^\pi = \pi \circ f|V$. Let $q \in V$ be a cusp point of a stable map $f_V^\pi : V \to \mathbb{R}^2$. For a sufficiently small neighborhood U of $f_V^\pi(q)$, the map $f_V^\pi|U' : U' \to U$ has degree ± 1, where U' is the component of $(f_V^\pi)^{-1}(U)$ containing q. If the degree of q is $+1$ (resp. -1), then we should paint q and $f_V^\pi(q)$ red (resp. blue).

For each $t \in \mathbb{R}$, a plane $\{(t, y.z) \in \mathbb{R}^3 |\ y, z \in \mathbb{R}\}$ is denoted by $\mathbb{R}_t^2$. Then, for almost all $t \in \mathbb{R}$, $f(V) \cap \mathbb{R}_t^2$ consists of immersed circles (or an empty set), $f(M_i) \cap \mathbb{R}_t^2$ consists of immersed surfaces (or an empty set) and $f(M_i) \cap \mathbb{R}_t^2$ is an extension of $f(V) \cap \mathbb{R}_t^2$. Therefore, from the pictures $f(M_1) \cap \mathbb{R}_{t_1}^2, f(M_1) \cap \mathbb{R}_{t_2}^2, \dots, f(M_1) \cap \mathbb{R}_{t_n}^2$ and $f(M_2) \cap \mathbb{R}_{t_1}^2, f(M_2) \cap \mathbb{R}_{t_2}^2, \dots, f(M_2) \cap \mathbb{R}_{t_n}^2$, we can see the immersed 3-dimensional manifold $f(M_1), f(M_2)$ and the image of the stable fold map $f(M)$. Note that the planes $\mathbb{R}_{t_1}^2, \mathbb{R}_{t_2}^2, \dots, \mathbb{R}_{t_n}^2$ can be chosen from the picture of the contour $f_V^\pi(S(f_V^\pi)) \subset \mathbb{R}^2$.

For a fold point $q \in S(f_V^\pi)$ of f_V^π, there exist local coordinates (u_1, u_2, u_3) and (v_1, v_2) around $q \in M$ and $\pi \circ f(q) \in \mathbb{R}^2$ such that

$$(v_1 \circ (\pi \circ f), v_2 \circ (\pi \circ f)) = (u_1, u_2^2 \pm u_3^2)$$

holds. Here, $S(f)$ corresponds to $\{u_3 = 0\}$. If q corresponds to the map $(v_1 \circ (\pi \circ f), v_2 \circ (\pi \circ f)) = (u_1, u_2^2 + u_3^2)$ (resp. $(v_1 \circ (\pi \circ f), v_2 \circ (\pi \circ f)) = (u_1, u_2^2 - u_3^2)$), then we should paint q and $\pi \circ f(q)$ red (resp. blue). From the local picture around $S(f_V^\pi)$, we have the following.

- On each connected component of $S(f_V^\pi) \setminus \{$cusp points$\}$, it should be colored by red or blue.
- If two connected components of $S(f_V^\pi) \setminus \{$cusp points$\}$ adjacent to the same cusp point, then they are painted by the different colors. See Figure 2 of the web version for example.

3. Construction of a stable fold map $f^{(2,1)} : L(2,1) \to \mathbb{R}^3$

In this section, we construct a stable fold map $f^{(2,1)} : L(2,1) \to \mathbb{R}^3$ such that $S(f^{(2,1)}) = T^2$ is a Heegaard surface of $L(2,1)$.

(Step 1.) Let $g : V \to \mathbb{R}^2$ be a stable map of a closed connected surface V to $\mathbb{R}^2$ such that the contour $g(S(g))$ and the inverse images $g^{-1}(\mathbb{R}_{t_1}) \cap V, \ldots, g^{-1}(\mathbb{R}_{t_{11}}) \cap V$ are depicted in Figure 1. Here, $\mathbb{R}_t$ is a line defined by $\mathbb{R}_t = \{(t, y) \in \mathbb{R}^2 | y \in \mathbb{R}\}$.

Since $g^{-1}(\mathbb{R}_{t_1}) \cap V, \ldots, g^{-1}(\mathbb{R}_{t_{11}}) \cap V$ can be seen as a sequence of immersed curves in $\mathbb{R}^2_{t_i}$, we can lift the stable map $g : V \to \mathbb{R}^2$ to a generic immersion $g' : V \to \mathbb{R}^3$ such that $g = \pi \circ g'$. From Figure 1, we can check that V is a torus. In the following, we consider that the sequence in Figure 1 is the sequence of immersed circles $g'(V) \cap \mathbb{R}^2_{t_1}, \ldots, g'(V) \cap \mathbb{R}^2_{t_{11}}$.

(Step 2.) From Figure 1, we construct two kinds of sequences of immersed surfaces which are extensions of immersed circles $g'(V) \cap \mathbb{R}^2_{t_1}, \ldots, g'(V) \cap \mathbb{R}^2_{t_{11}}$. Figure 2 represents one sequence of immersed surfaces and Figure 3 represents another sequence.

By combining the immersed surfaces in Figure 2, we have an immersion $f_1 : M_1 \to \mathbb{R}^3$ which is one extension of the generic immersion $g' : V \to \mathbb{R}^3$. Also, by combining the immersed surfaces in Figure 3, we have an immersion $f_2 : M_2 \to \mathbb{R}^3$ which is another extension of the generic immersion $g' : V \to \mathbb{R}^3$. We define the orientation of M_1 (resp. M_2) so as the immersion f_1 (resp. f_2) is an orientation preserving (resp. orientation reversing). In Figure 2 (resp. Figure 3), green bands explain how each immersed surface $f_1(M_1) \cap \mathbb{R}^2_{t_i}$ (resp. $f_2(M_2) \cap \mathbb{R}^2_{t_i}$) is obtained as the extension of the immersed circles $g'(V) \cap \mathbb{R}^2_{t_i}$. See the web version.

(Step 3) Let $C \subset V$ be a circle such that $C \subset S(g)$ and the image $g(C)$ is depicted as gray thick lines in Figure 4. By a regular homotopy of f_2, we can check that M_2 is a solid torus and C is a meridian circle of M_2. By a regular homotopy of f_1, we can check that M_1 is a solid torus and C is a $(2,1)$-curve of M_1. That is, C turns twice in the longitude direction and once in the the meridian direction on M_1. Therefore, by attaching these immersions f_1 and f_2, we obtain a stable fold map $f^{(2,1)} = f_1 \cup f_2 : M_1 \cup_V M_2 = L(2,1) \to \mathbb{R}^3$ such that $S(f^{(2,1)}) = V = T^2$ is a Heegaard surface.

4. Construction of a stable fold map $f^{(p,1)} : L(p,1) \to \mathbb{R}^3$

In this section, we construct a stable fold map $f^{(p,1)} : L(p,1) \to \mathbb{R}^3$ such that $S(f^{(p,1)}) = T^2$ is a Heegaard surface of $L(p,1)$ $(p \geq 2)$.

(Step 1.) Let $g' : V \to \mathbb{R}^3$ be a generic immersion of a closed connected surface V to $\mathbb{R}^3$ such that $g = \pi \circ g'$ is a stable map and the contour $g(S(g))$ is depicted in Figure 5. Let U be a subset of $\mathbb{R}^2$ depicted in Figure 5. The image $g(V) \cap (\mathbb{R}^2 \setminus U)$ of Figure 5 is the same as that of Figure 1. Therefore, in Figure 6, we only describe a sequence of immersed arcs $g'(V) \cap \pi^{-1}(\mathbb{R}_t \cap U)$. From Figures 5 and 6, we can check that V is a torus.

(Step 2.) From Figure 6, we construct two kinds of sequences of immersed surfaces which are extensions of immersed arcs $g'(V) \cap \pi^{-1}(\mathbb{R}_t \cap U)$. Figure 7 represents one sequence of immersed surfaces and Figures 8 represents another sequence. By combining the immersed surfaces in Figure 7, we have

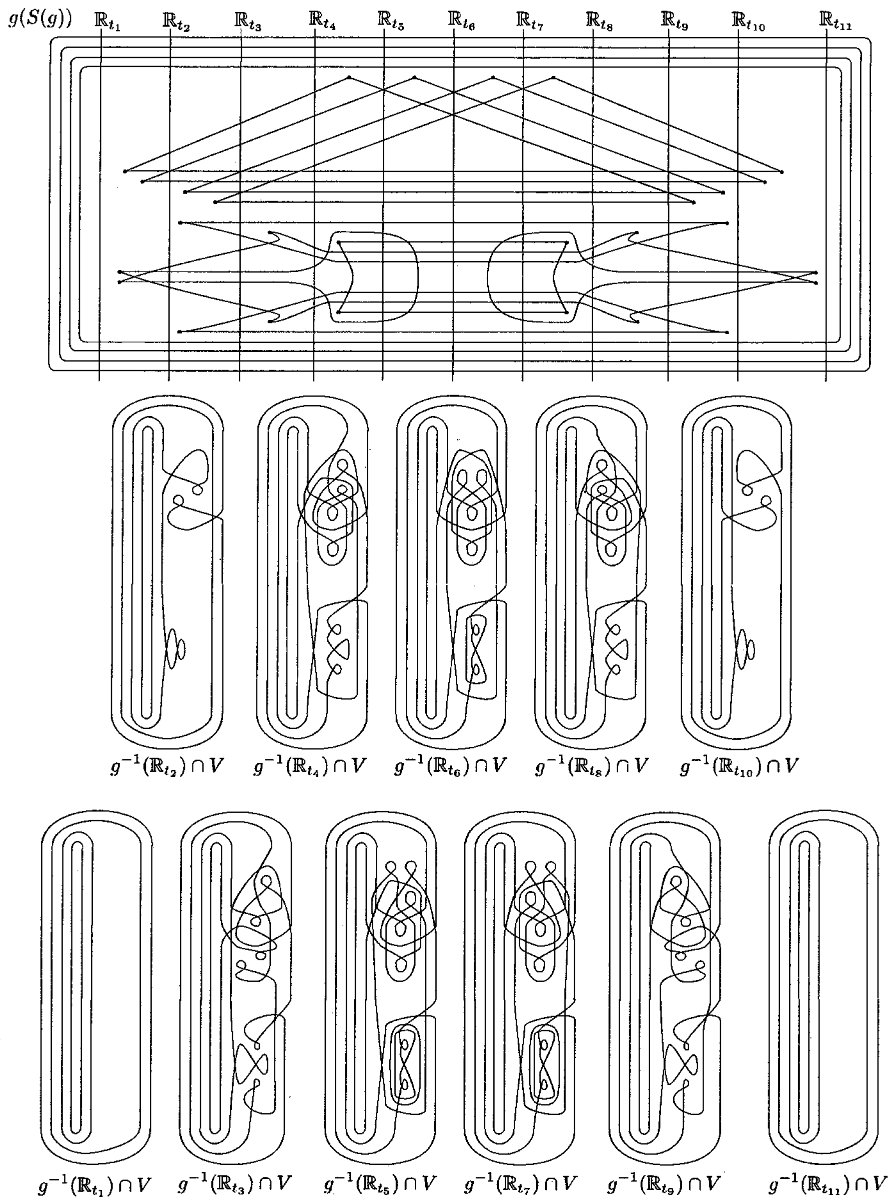

FIGURE 1. The contour of $g : V \to \mathbb{R}^2$ and the sequence of sectional faces of $g(V)$ or $g'(V)$.

an immersion $f_1 : M_1 \to \mathbb{R}^3$ which is one extension of the generic immersion $g' : V \to \mathbb{R}^3$. Also, by combining the immersed surfaces in Figure 8, we have an immersion $f_2 : M_2 \to \mathbb{R}^3$ which is another extension of the generic immersion $g' : V \to \mathbb{R}^3$. We define the orientation of M_1 (resp. M_2) so as the immersion f_1 (resp. f_2) is an orientation preserving (resp. orientation reversing). In Figure 7 (resp. Figure 8), green bands explain how each immersed

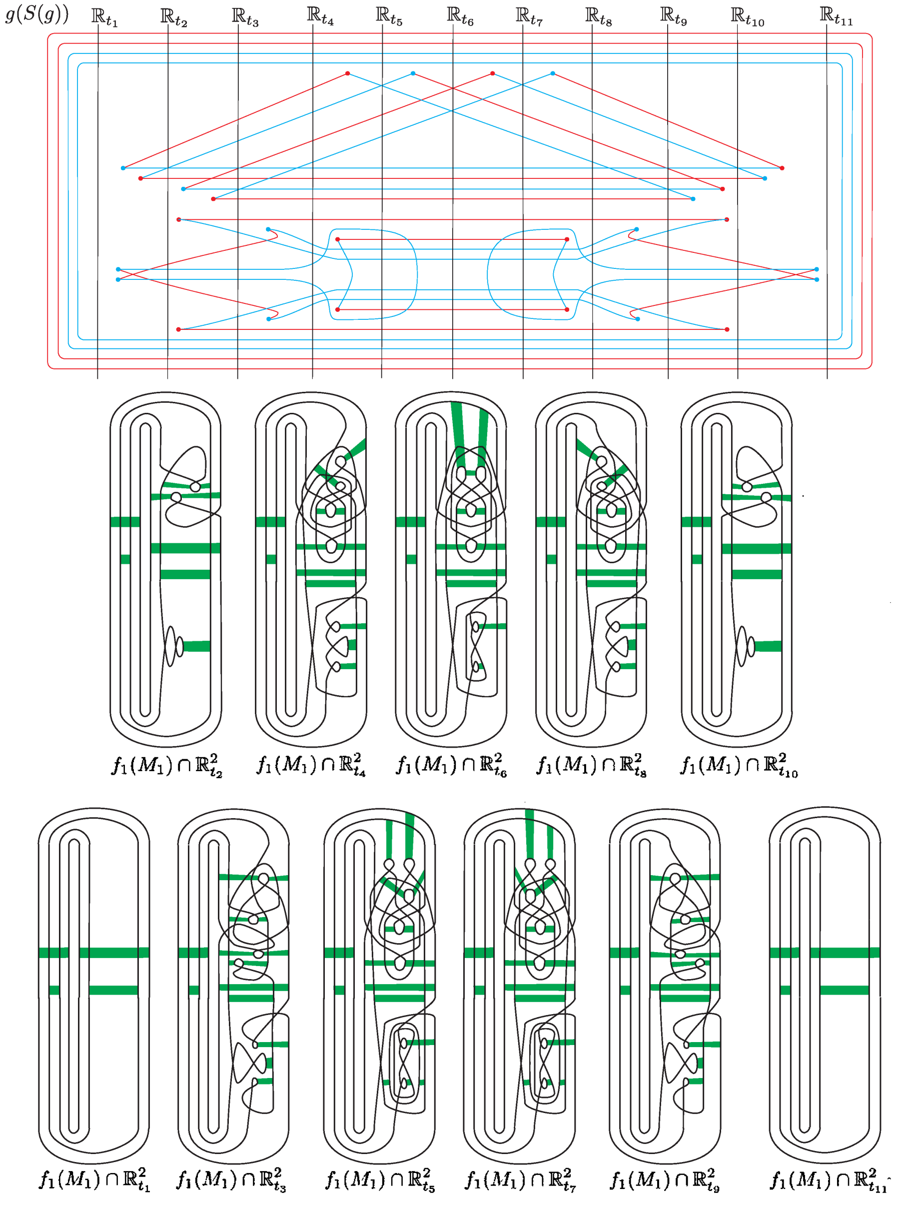

FIGURE 2. The sequence of sectional faces of $f_1(M_1)$.

surface $f_1(M_1)\cap\pi^{-1}(\mathbb{R}_t\cap U)$ (resp. $f_2(M_2)\cap\pi^{-1}(\mathbb{R}_t\cap U)$) is obtained as the extension of the immersed arcs $g'(V)\cap\pi^{-1}(\mathbb{R}_t\cap U)$. See the web version.

(Step 3) Let $C\subset V$ be a circle such that $C\subset S(g)$ and the image $g(C)$ is depicted as gray thick lines in Figure 9. By a regular homotopy of f_2, we can check that M_2 is a solid torus and C is a meridian circle of M_2. By a regular homotopy of f_1, we can check that M_1 is a solid torus and C is a $(p,p-1)$-curve of M_1. Therefore, by attaching these immersions f_1 and

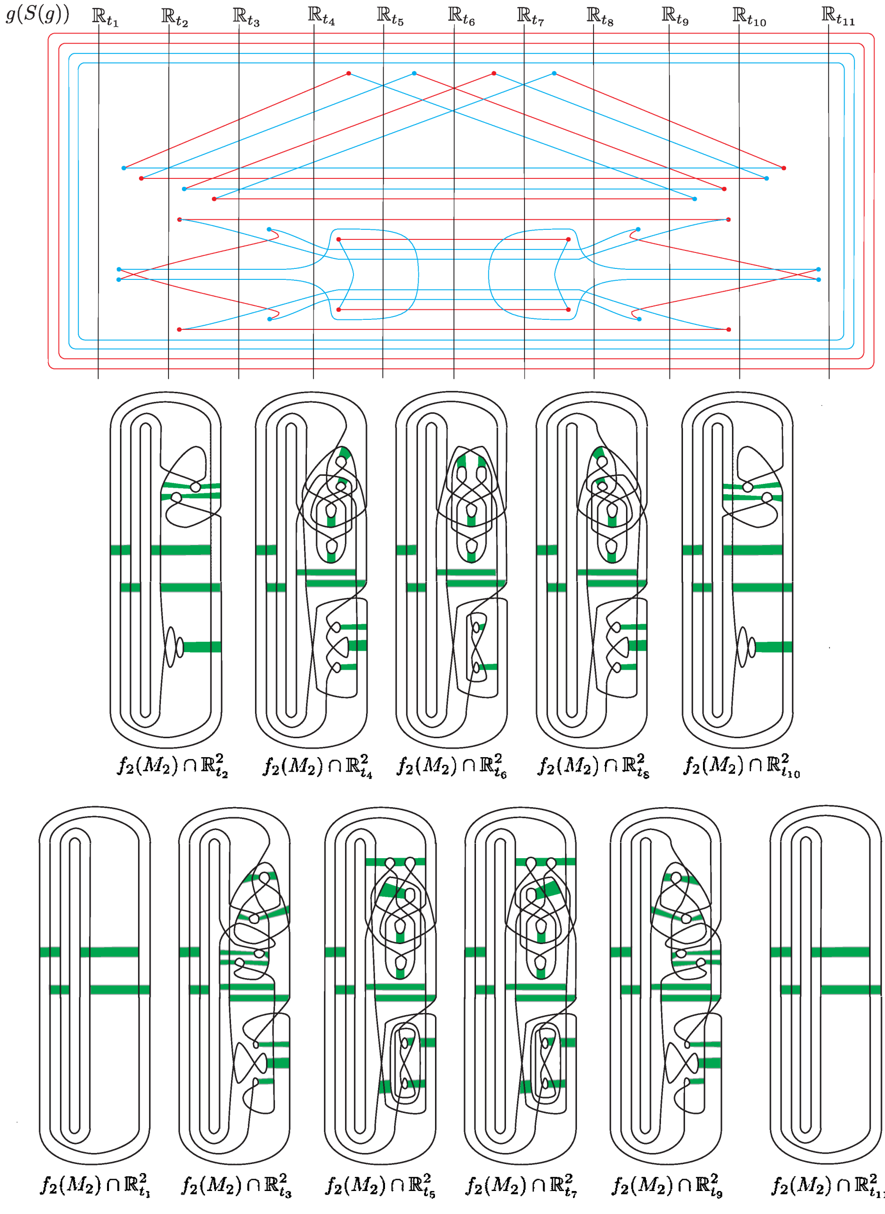

FIGURE 3. The sequence of sectional faces of $f_2(M_2)$.

f_2, we obtain a stable fold map $f_1 \cup f_2 : M_1 \cup_V M_2 = L(p, p-1) \to \mathbb{R}^3$ such that $S(f_1 \cup f_2) = V = T^2$ is a Heegaard surface. Since $L(p, p-1)$ is diffeomorphic to $L(p, 1)$, $f^{(p,1)} = f_1 \cup f_2$ is a desired stable fold map.

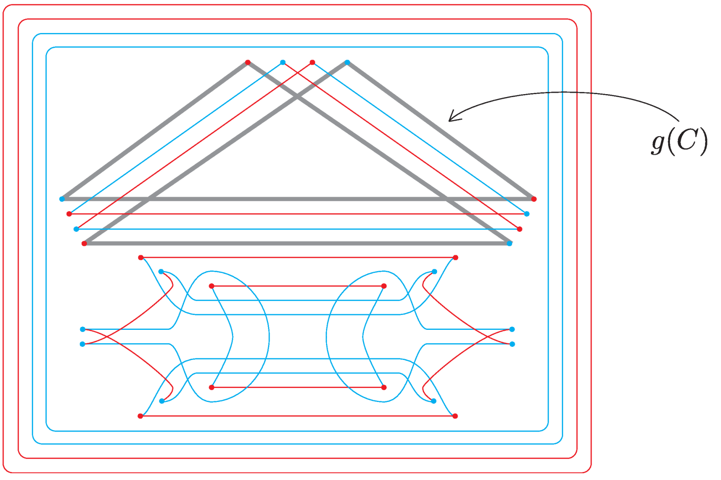

FIGURE 4. The image of the curve C which is a meridian circle of M_2.

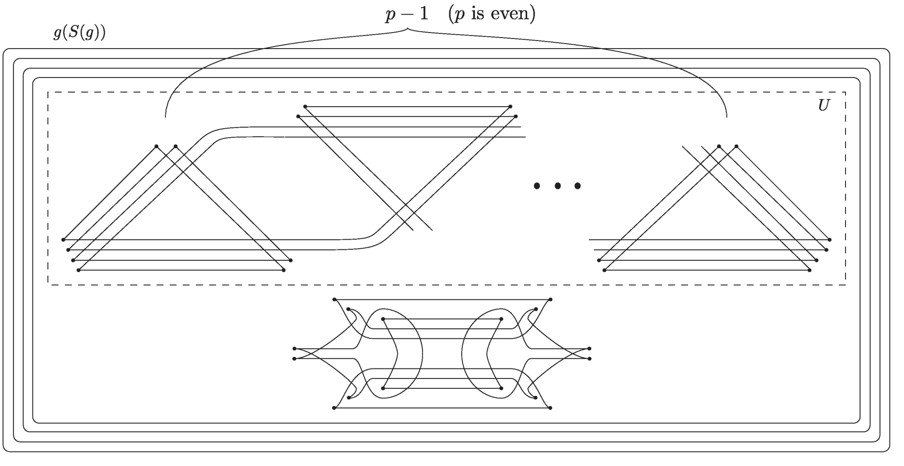

FIGURE 5. The contour of $g : V \to \mathbb{R}^2$.

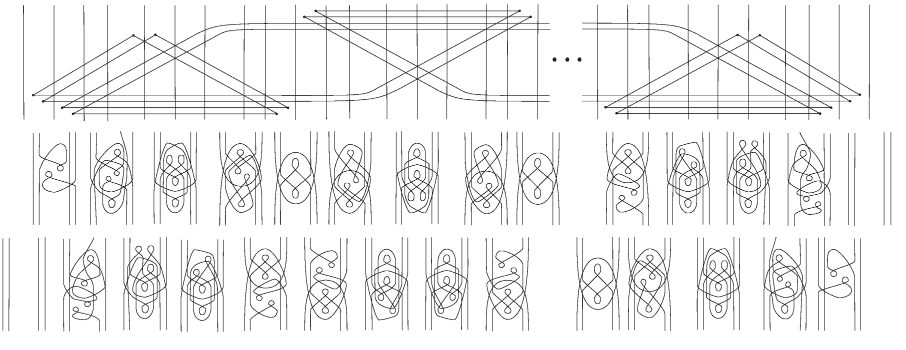

FIGURE 6. The sequence of the sectional faces of $g'(V)$.

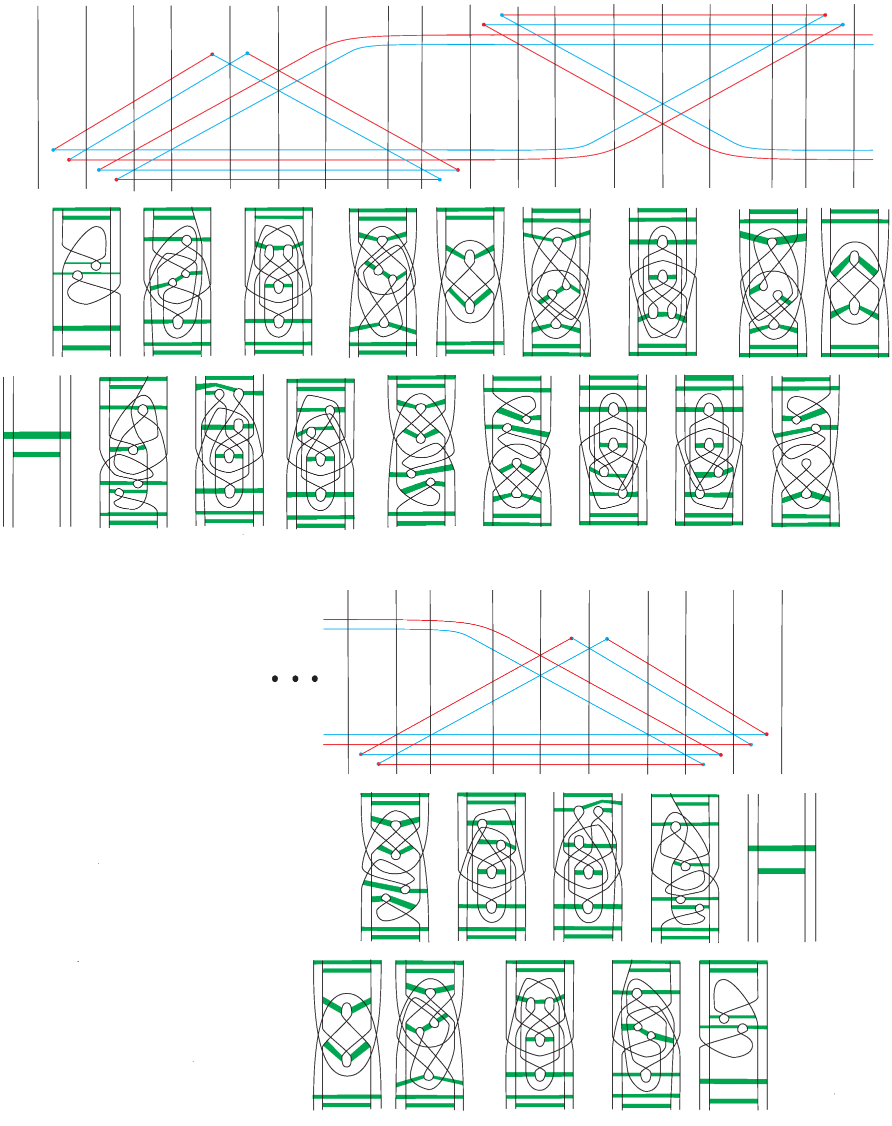

FIGURE 7. The sequence of sectional faces of $f_1(M_1)$.

5. REMARKS AND PROBLEMS

In Sections 3 and 4, we only construct a stable fold map of $L(p,1)$ whose singular set is a genus one Heegaard surface. Therefore, we have a following problem.

Problem 5.1. Construct a stable fold map $f^{(p,q)} : L(p,q) \to \mathbb{R}^3$ such that $S(f^{(p,q)})$ is a genus one Heegaard surface $(p-1 > q > 1)$.

For the stable fold map $f^{(2,1)} : L(2,1) \to \mathbb{R}^3$ of Section 3, we can check that $(f^{(2,1)})^{-1}\left(f^{(2,1)}(L(2,1)) \cap \mathbb{R}^2_{t_6}\right)$ is a torus in $L(2,1)$. Let $\mathbb{R}^3_{(-\infty,t_6]}$ and

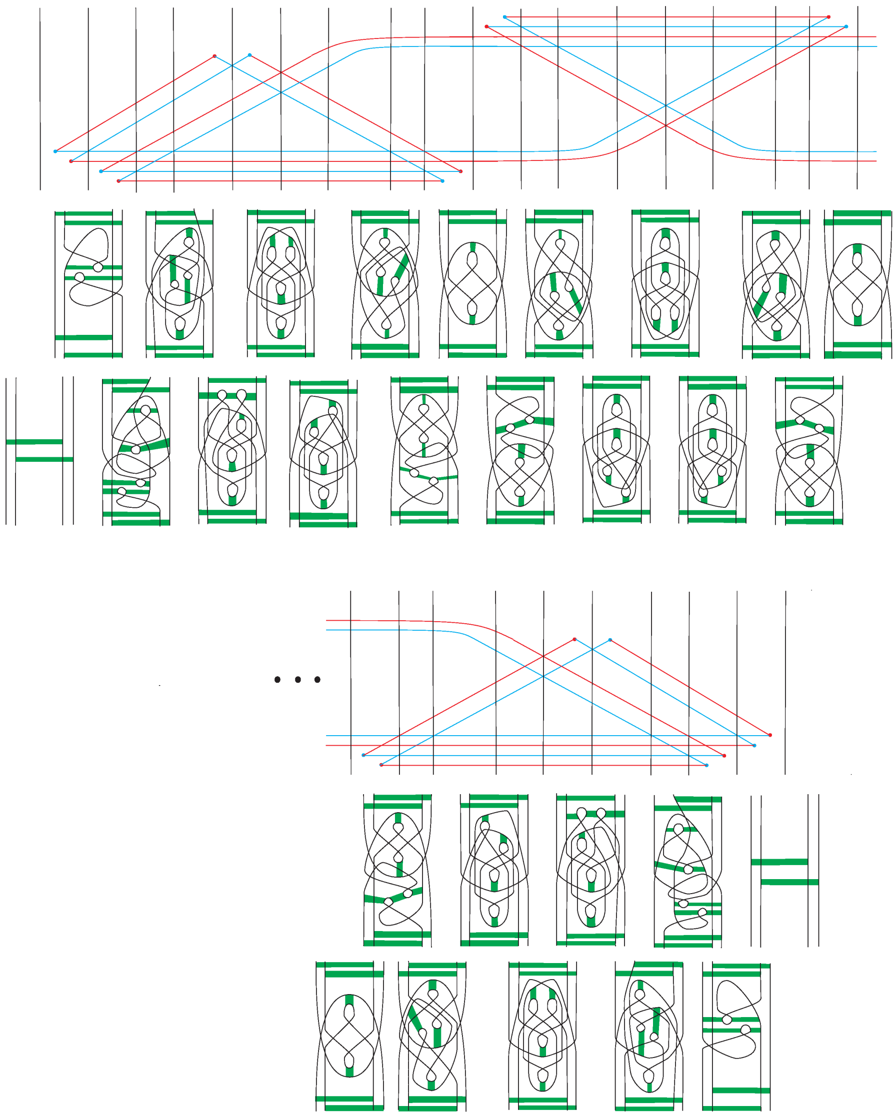

FIGURE 8. The sequence of sectional faces of $f_2(M_2)$.

$\mathbb{R}^3_{[t_6,\infty)}$ be half spaces defined by $\mathbb{R}^3_{(-\infty,t_6]} = \{(x,y,z) \in \mathbb{R}^3 \mid x \in (-\infty, t_6]\}$ and $\mathbb{R}^3_{[t_6,\infty)} = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [t_6,\infty)\}$. Let N_1 and N_2 be submanifolds of $L(2,1)$ defined by $N_1 = L(2,1) \cap (f^{(2,1)})^{-1}(\mathbb{R}^3_{(-\infty,t_6]})$ and $N_2 = L(2,1) \cap (f^{(2,1)})^{-1}(\mathbb{R}^3_{[t_6,\infty,t_6)})$. We have a following problem.

Problem 5.2. Does the decomposition $N_1 \cup_{T^2} N_2$ represent a genus one Heegaard splitting of $L(2,1)$?

Let $S^3 = D^3_1 \cup_{S^2_1} S^2 \times I \cup_{S^2_2} D^3_2$ be a decomposition of S^3 and $e : S^3 \to \mathbb{R}^3$ be a stable fold map such that $S(e) = S^2_1 \cup S^2_2$ and $e|D^3_1$ and $e|D^3_2$ are orientation

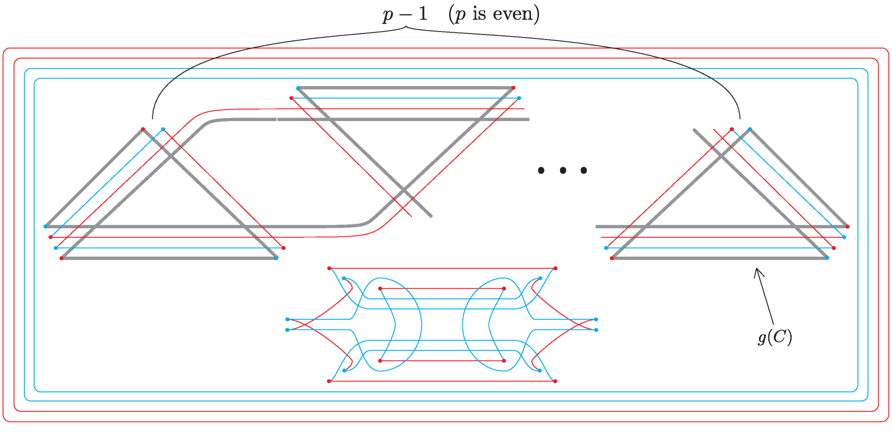

FIGURE 9. The image of the curve C which is a meridian circle of M_2.

preserving immersions and $e|S^2 \times I$ is an orientation reversing immersion. Figure 10 represents the contour of the stable map $e^{\pi}_{S^2_1 \cup S^2_2} : S^2_1 \cup S^2_2 \to \mathbb{R}^2$ and the sequence of the sectional faces of $e(S^2_1 \cup S^2_2)$. Figure 11 (resp. Figure 12) represents the sequence of the sectional faces of $e(D^3_1)$ (resp. $e(D^3_2)$ and Figure 13 represents the sequence of the sectional faces of $e(S^2 \times I)$.

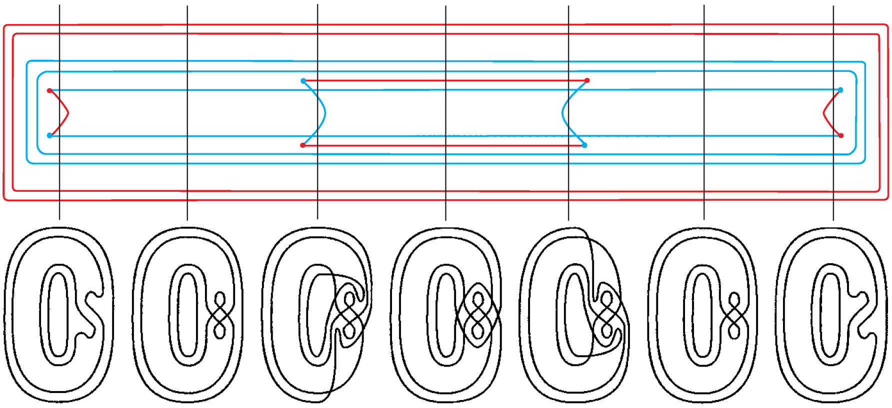

FIGURE 10. The contour of $e^{\pi}_{S^2_1 \cup S^2_2} : S^2_1 \cup S^2_2 \to \mathbb{R}^2$ and the sequence of the sectional faces of $e(S^2_1 \cup S^2_2)$.

By a connected sum of the two stable fold maps $f^{(p,1)} \sharp e$ and the Eliashberg's trick which is introduced in [2], we have a stable fold map $f_2^{(p,1)}$: $L(p,1) \to \mathbb{R}^3$ such that $S(f_2^{(p,1)}) = T^2 \sharp T^2$ is a genus two Heegaard surface ($p \geq 2$). The contour of $\pi \circ f_2^{(p,1)}|S(f_2^{(p,1)})$ is depicted in Figure 14. By repeating the above operation, we have a stable fold map $f_k^{(p,1)} : L(p,1) \to \mathbb{R}^3$ such that $S(f_k^{(p,1)}) = \overset{k}{\sharp} T^2$ is a genus k Heegaard surface ($p \geq 2$).

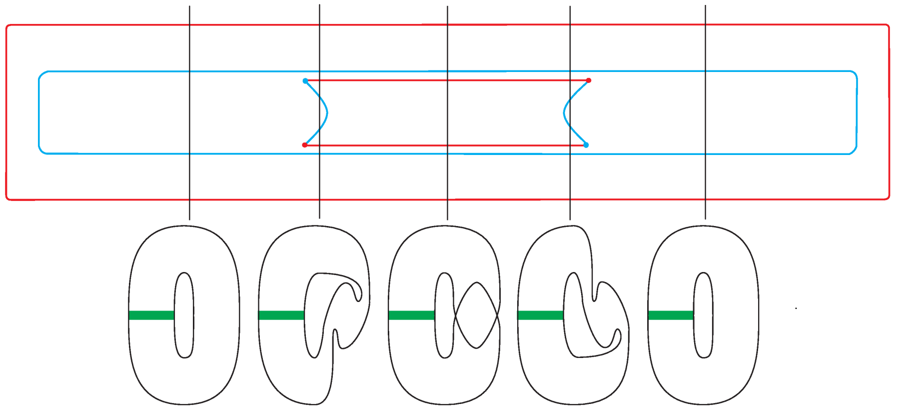

FIGURE 11. The sequence of the sectional faces of $e(D^3_1)$.

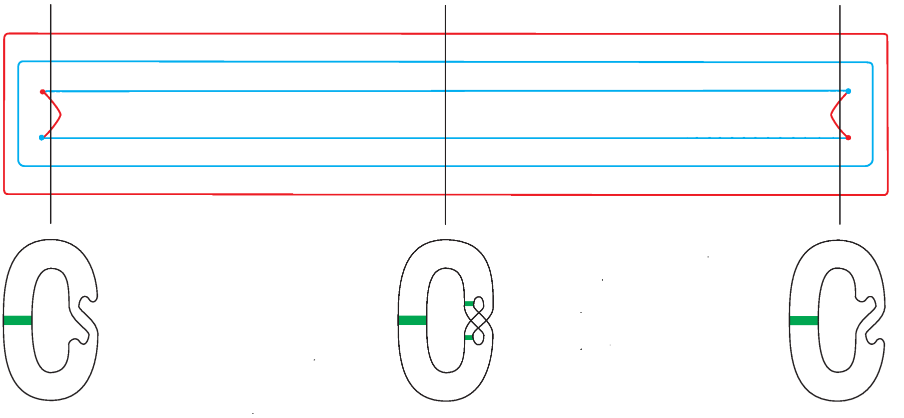

FIGURE 12. The sequence of the sectional faces of $e(D^3_2)$.

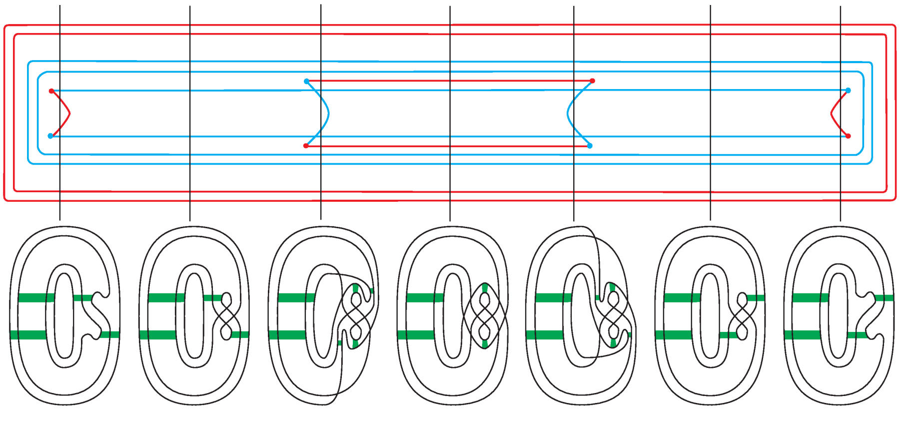

FIGURE 13. The sequence of the sectional faces of $e(S^2 \times I)$.

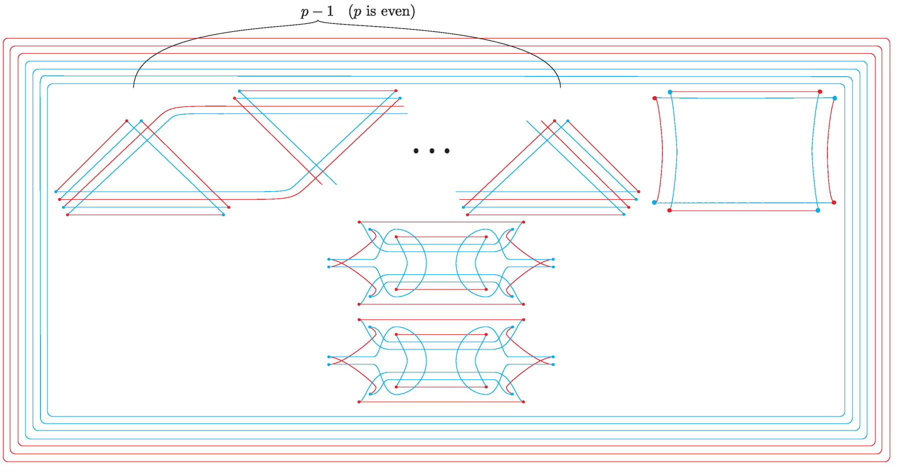

FIGURE 14. The contour of $\pi \circ f_2^{(p,1)}|S(f_2^{(p,1)})$.

If we use the Eliashberg's trick for the stable fold map $e : S^3 \to \mathbb{R}^3$, we have a stable fold map $f^{(1,0)} : S^3 \to \mathbb{R}^3$ such that $S(f^{(1,0)}) = T^2$ is a genus one Heegaard surface. Therefore, we also have a stable fold map $f_k^{(1,0)} : S^3 \to \mathbb{R}^3$ such that $S(f_k^{(1,0)}) = \overset{k}{\sharp} T^2$ is a genus k Heegaard surface. We have a following problem.

Problem 5.3. Construct a nontrivial stable fold map $f : L(p, p-1) \to \mathbb{R}^3$ such that $S(f)$ is a genus k Heegaard surface ($p \geq 1, k \geq 2$).

Let $\mathrm{SI}(3,1)$ be the group of oriented bordism classes of immersions of closed oriented 3-dimensional manifolds in $\mathbb{R}^4$ and $\mathrm{SFold}(3,0)$ the group of oriented fold cobordism classes of fold maps of closed oriented 3-dimensional manifolds into $\mathbb{R}^3$. Let $K : S^3 \to \mathbb{R}^4$ be an immersion which is constructed from the track of the standard Froissart-Morin's eversion $S^2 \times I \to \mathbb{R}^4$. Hughes [5] showed that the immersion K is a generator of $\mathrm{SI}(3,1)$. Hirato-Takase [4] showed that the homomorphisim $\mathfrak{m} : \mathrm{SFold}(3,0) \to \mathrm{SI}(3,1)$ is an isomorphism. Since we can check that e and $f^{(1,0)} : S^3 \to \mathbb{R}^3$ are oriented fold cobordant, and that the bordism class of K is equal to $\mathfrak{m}(e)$, the stable fold map $f^{(1,0)} : S^3 \to \mathbb{R}^3$ is a generator of $\mathrm{SFold}(3,0)$. This also shows that $f^{(1,0)} : S^3 \to \mathbb{R}^3$ is a generator of the third stable stem π_3^S.

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